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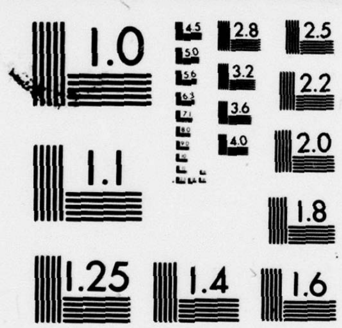


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# CONTINUITY OF BEST RECIPROCAL POLYNOMIAL APPROXIMATION ON $[0, \infty)$

by

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LEVEL







## Introduction

Over the past ten years considerable progress has been made in studying various questions concerning rational approximation on unbounded sets. To a large extent the starting point of this effort was the paper of Cody, Meinardus and Varga [6] and this has led to investigations of best approximation properties in various settings [1-4, 8-9] and studies of the error of best approximation [10-12].

In this paper we wish to study the best approximation properties of strong uniqueness and continuity of the best approximation operator for reciprocal polynomial approximation on $[0, \infty)$ of continuous positive functions tending to 0 as $x \to \infty$. Thus, we define

$$(1) \qquad C_0^+[0, \infty) = \{f \in C[0, \infty): f(x) > 0,\ x \in [0, \infty) \text{ and } \lim_{x\to\infty} f(x) = 0\},$$

and

$$(2) \qquad R_n = \{\tfrac{1}{p}: p \in \Pi_n,\ p(x) > 0,\ x \in [0, \infty)\},\ n \geq 1,$$

where $\Pi_n$ denotes the class of all algebraic real polynomials of degree $\leq n$. Furthermore, define $\|f\| = \sup\{|f(x)|: x \in [0, \infty)\}$ in what follows. In this setting, it is known that best approximations exist and are unique [3, 4] and that the following characterization theorem holds:

[1]Research sponsored by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant AFOSR-76-2878C and by National Science Foundation under Grant MCS-78-05847.

THEOREM 1 [4]. Let $f \in C_0^+[0, \infty) \sim R_n$ with $n \geq 1$. Then $\frac{1}{p^*}$ is the best approximation to f from $R_n$ on $[0, \infty)$ iff

(i) (standard alternation) there exist $\{x_i\}_{i=0}^{n+1}$, $0 \leq x_0 < x_1 < \ldots < x_{n+1}$, such that $|f(x_i) - \frac{1}{p^*(x_i)}| = \|f - \frac{1}{p^*}\|$, $i = 0, \ldots, n + 1$ and

$$f(x_i) - \frac{1}{p^*(x_i)} = -(f(x_{i+1}) - \frac{1}{p^*(x_{i+1})}),\ i = 0, \ldots, n;$$

or

(ii) (nonstandard alternation) $\partial p^* \leq n - 1$ and there exist $\{x_i\}_{i=0}^{n}$,

$$0 \leq x_0 < x_1 < \ldots < x_n \text{ such that } f(x_i) - \frac{1}{p^*(x_i)} = (-1)^{n-i} \|f - \frac{1}{p^*}\| .$$

In both cases the points $\{x_i\}$ are called extreme points. Also, we wish to note the for $n \geq 1$, $p^*$ cannot be a constant. Indeed, since $f(x) > 0$ for all $x \in [0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then in order for the reciprocal of a constant, $\frac{1}{c^*}$, to be a best approximation to f, we must have that $c^* = \frac{2}{M}$ where $M = \max_{x\geq 0} f(x)$. Since $f(x) \to 0$ as $x \to \infty$ we can find $x_0 > 0$ such that $f(x) < M$ for $x \geq x_0$. It is then easilty seen that for $p^*(x) = \varepsilon(x - x_0) + c^*$ with $\varepsilon > 0$ and sufficiently small that $\|f - \frac{1}{p^*}\| < \|f - \frac{1}{c^*}\|$ by a straightforward continuity-compactness argument.

In addition, it has been shown in [3] that if $\frac{1}{p^*} \in R_n$ is the best approximation to $f \in C_0^+[0, \infty)$ from $R_n$ with $\partial p^* = n$ then both strong uniqueness (i.e., $\|f - \frac{1}{p}\| - \|f - \frac{1}{p^*}\| \geq \gamma \|\frac{1}{p} - \frac{1}{p^*}\|$, $\gamma = \gamma(f) > 0$ for all $\frac{1}{p} \in R_n$) and Lipschitz continuity of the best approximation operator at f (i.e., $\|\frac{1}{p^*} - \frac{1}{p_g}\| \leq \beta \|f - g\|$, $\beta = \beta(f) > 0$, $g \in C_0^+[0, \infty)$ and $\frac{1}{p_g}$ the best approximation to g from $R_n$) hold. Furthermore, it was shown in [3] that for each f whose corresponding best approximation from $R_n$, $\frac{1}{p^*}$, satisfies $\partial p^* \leq n - 2$ the strong uniqueness theorem cannot hold. In this present paper we shall prove that if $f \in C_0^+[0, \infty)$ has $\frac{1}{p^*} \in R_n$ as its best approximation then (i) if $\partial p^* \leq n - 2$ (i.e. $\frac{1}{p^*}$ is deficient of order two or more) then the best approximation operator is discontinuous at f and (ii) if $\partial p^* = n - 1$ then the best approximation operator is continuous at f. It remains open as to whether or not a strong uniqueness theorem holds in the case that $\partial p^* = n - 1$.

## Main Results

In this section we state and prove our main results. The first result establishing the discontinuity of the best approximation operator is given in two parts. The first theorem will treat this problem for the case that either $\frac{1}{p^*}$ is deficient of order three or more, or when $\frac{1}{p^*}$ is deficient of order two and $f - \frac{1}{p^*}$ possesses a standard alternating sequence. In this case we can prove even stronger results concerning the discontinuous behavior of the best approximation operator. The second theorem will treat the discontinuity of the best approximation operator when $\frac{1}{p^*}$ is deficient of order two with only nonstandard alternation holding for $f - \frac{1}{p^*}$. Our final result will be to prove that the best approximation operator is continuous whenever $\frac{1}{p^*}$ is deficient of order one.

THEOREM 2. Let $f \in C_0^+[0, \infty) \sim R_n$ and $\frac{1}{p^*} \in R_n$ be the best approximation to f from $R_n$. Further, assume that $\partial p^* \leq n - 2$ and that if $\partial p^* = n - 2$

then $f - \frac{1}{p^*}$ possesses a standard alternating set. Then, given $\varepsilon > 0$ there exists $\delta > 0$, $\left\{\frac{1}{p_k}\right\}_{k=1}^{\infty} \subset R_n$ and $\{g_k\}_{k=1}^{\infty} \subset C_0^+[0, \infty)$ such that each $g_k$ has $\frac{1}{p_k}$ as its best approximation from $R_n$, $g_k$ converges uniformly to $f$ on $[0, \infty)$ and $\delta \leq \|\frac{1}{p^*} - \frac{1}{p_k}\| \leq \varepsilon$ for all $k$.

<u>REMARK</u>. This theorem establishes that not only is the best approximation operator discontinuous at $f$ but, in fact, that it is also not possible for a local (relative to $\frac{1}{p^*}$) strong uniqueness result to hold.

<u>Proof</u>. Set $E = \|f - \frac{1}{p^*}\| > 0$ and assume without loss of generality that $\varepsilon \leq \frac{E}{4}$. Set $\delta = \frac{\varepsilon}{8}$. Since we are assuming throughout this paper that $n \geq 1$, we have that $p^*(x)$ is not identically equal to a constant which implies that $\lim_{x\to\infty} p^*(x) = \infty$. Select $\beta > 0$ such that $f(x) \leq \varepsilon$ and $p^*(x) > \frac{4}{\varepsilon}$ for all $x \geq \beta$. Set $e_k = (\frac{\varepsilon}{4} + \frac{2}{p^*(k)})^{-1}$ and note that for $k \geq \beta$, $p^*(k) > e_k$. Define $p_k$ $\Pi_n$ by

$$p_k(x) = e_k + (p^*(x) - e_k)[(\frac{x}{k} - 1)^2 + \frac{e_k}{p^*(k) - e_k}], \; k \geq \beta.$$

Since for all $x \geq \beta, p^*(x) > e_k$, we have that $(p^*(x) - e_k)[(\frac{x}{k} - 1)^2 + \frac{e_k}{p^*(k)-e_k}] > 0$ implying that $p_k(x) > e_k > \frac{4}{3\varepsilon}$ for $x \geq \beta$.

Next, observe that $e_k \to \frac{1}{\varepsilon}$ as $k \to \infty$ and $(\frac{x}{k} - 1)^2$ converges uniformly to 1 on $[0, \beta]$ as $k \to \infty$. Thus, $p_k$ converges uniformly to $p^*$ on $[0, \beta]$ as $k \to \infty$. Now, let

$$\eta = \min\left( \min_{x\in[0,\beta]} f(x), \min_{x\in[0,\beta]} \frac{1}{p^*(x)}, \varepsilon\right)$$

and select $\mu \geq \beta$ such that for $k \geq \mu$, $\max_{x\in[0,\beta]} \left|\frac{1}{p_k(x)} - \frac{1}{p^*(x)}\right| \leq \frac{\eta}{2}$. This implies that $\frac{1}{p_k(x)} \geq \frac{1}{p^*(x)} - \frac{\eta}{2} \geq \frac{\eta}{2} > 0$ for all $x \in [0, \beta]$ and $k \geq \mu$. Hence $\frac{1}{p_k} \in R_n$ for $k \geq \mu$ as $p_k(x) > 0$ for all $x \in [0, \infty)$.

Now, since $p_k(k) = 2e_k$ we have that $\frac{1}{p_k(k)} - \frac{1}{p^*(k)} = \frac{\varepsilon}{8} = \delta$. Thus, $\|\frac{1}{p_k} - \frac{1}{p^*}\| \geq \delta$ for all $k \geq \mu$. Also, for $k \geq \mu$ we have that $\left|\frac{1}{p_k(x)} - \frac{1}{p^*(x)}\right| \leq \frac{\eta}{2} < \varepsilon$ for $x \in [0, \beta]$. In addition, for $x \geq \beta$ we have that $\frac{1}{p^*(x)} < \frac{\varepsilon}{4}$ and $\frac{1}{p_k(x)} \leq \frac{3\varepsilon}{4}$ for $k \geq \mu$. Hence $\|\frac{1}{p_k} - \frac{1}{p^*}\| \leq \varepsilon$ as claimed.

Finally, define $g_k$, for $k \geq \mu$, by

$$g_k(x) = \begin{cases} f(x) + \frac{1}{p_k(x)} - \frac{1}{p^*(x)}, & x \in [0, \beta] \\ f(x), & x \geq \beta + \frac{1}{k} \\ \text{linear with endpoint values } f(\beta) + \frac{1}{p_k(\beta)} - \frac{1}{p^*(\beta)} \text{ and } f(\beta+\frac{1}{k}), & x \quad [\beta,\beta+\frac{1}{k}]. \end{cases}$$

Clearly, $g_k \in C[0, \infty)$, $k \geq \mu$ and since $f(x) + \frac{1}{p_k(x)} - \frac{1}{p^*(x)} \geq f(x) - \frac{\eta}{2} > 0$ for $x \in [0, \beta]$ we have that $g_k \in C_0^+[0, \infty)$ for $k \geq \mu$. Since $f(\beta)+\frac{1}{p_k(\beta)}-\frac{1}{p^*(\beta)}$ $< f(\beta) + \frac{\eta}{2} \leq f(\beta) + \frac{\varepsilon}{2} \leq f(\beta) + \frac{E}{8} \leq \frac{3}{8}E$ and $f(\beta + \frac{1}{k}) \leq \frac{E}{4}$ we have that

$g_k(x) \leq \frac{3}{8}E$ for $x \in [\beta, \beta + \frac{1}{k}]$ implying $g_k(x) \leq \frac{3E}{8}$ for $x \geq \beta$. Also, $\|\frac{1}{p_k} - \frac{1}{p^*}\| < \varepsilon \leq \frac{E}{4}$ implies that $\frac{1}{p_k(x)} < \frac{1}{p^*(x)} + \frac{E}{4} \leq \frac{E}{2}$ for $x \geq \beta$. From this it follows that $\left|g_k(x) - \frac{1}{p_k(x)}\right| \leq \frac{7}{8}E$ for $x \geq \beta$. In addition, for for $x \in [0, \beta]$ we have that $g_k(x) - \frac{1}{p_k(x)} = f(x) - \frac{1}{p^*(x)}$ and this implies that $g - \frac{1}{p_k}$ exhibits the same alternating behavior as $f - \frac{1}{p^*}$ on $[0, \infty)$. Thus, if $f - \frac{1}{p^*}$ has a standard alternating set so does $g_k - \frac{1}{p_k}$ implying that $\frac{1}{p_k}$ is the best approximation to $g_k$ from $R_n$ on $[0, \infty)$. If $f - \frac{1}{p^*}$ possesses only a nonstandard alternating set then so does $g_k - \frac{1}{p_k}$. Since in this case we must have that $\partial p_k \leq n - 3$, we must have that $\partial p_k \leq n - 1$ implying once again that $\frac{1}{p_k}$ is the best approximation to $g_k$ from $R_n$ on $[0, \infty)$. Since it is clear that $g_k$ converges uniformly to f on $[0, \infty)$, the proof is completed by relabeling the sequences $\left\{\frac{1}{p_k}\right\}_{k=\mu}^{\infty}$ and $\{g_k\}_{k=\mu}^{\infty}$ as $\left\{\frac{1}{p_k}\right\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$, respectively. ∎

For the case that $\partial p^* = n - 2$ and $f - \frac{1}{p^*}$ has only a nonstandard alternating sequence we have the slightly weaker theorem:

<u>THEOREM 3</u>. Let $f \in C_0^+[0, \infty) \sim R_n$ and $\frac{1}{p^*} \in R_n$ be its best approximation from $R_n$. Further, assume that $\partial p^* = n - 2$ and $f - \frac{1}{p^*}$ possesses only a nonstandard alternating set. Then there exists $\left\{\frac{1}{p_k}\right\}_{k=1}^{\infty} \subset R_n$ and $\{g_k\}_{k=1}^{\infty} \subset C_0^+[0, \infty)$ such that for each k, $\frac{1}{p_k}$ is the best approximation to $g_k$ from $R_n$ on $[0, \infty)$, $g_k$ converges uniformly to f on $[0, \infty)$ and $\|\frac{1}{p_k} - \frac{1}{p^*}\| \geq \frac{7}{8}E$ where $E = \|f - \frac{1}{p^*}\| > 0$.

Proof. Select $\beta > 0$ such that $p^*(x) \geq \frac{8}{E}$, $f(x) \leq \frac{E}{8}$ and $p^*(x)$ is monotone increasing for $x \geq \beta$. For $k \geq \beta$, define

$$p_k(x, t) = t + (p^*(x) - t)[(\frac{x}{k} - 1)^2 + \frac{t}{p^*(k) - t}],\ 0 \leq t \leq \frac{1}{E},\ x \geq 0.$$

Note that $p_k(x, t)$ is continuous on $[0, \infty) \times [0, \frac{1}{E}]$. Define $h(t) = \min\{p_k(x, t): x \in [\beta, 2k]\}$ and observe that $h$ is a continuous function of $t$, $0 \leq t \leq \frac{1}{E}$. In addition, $h(0) = \min\{p^*(x)(\frac{x}{k} - 1)^2: x \in [\beta, 2k]\} = 0$ as $k \geq \beta$ and that $h(\frac{1}{E}) = \min\{\frac{1}{E} + (p^*(x) - \frac{1}{E})[(\frac{x}{k} - 1)^2 + \frac{1}{Ep^*(k) - 1}]: x \in [\beta,2k]\} > \frac{1}{E}$ as $p^*(x) > \frac{1}{E}$ on $[\beta, 2k]$. Select $e_k \in (0, \frac{1}{E})$ so that $h(e_k) = \frac{1}{E}$. Thus, $p_k(x, e_k) \geq \frac{1}{E} > 0$ for $x \in [\beta, 2k]$. Observe that $p_k(x, e_k)$ converges uniformly to $p^*(x)$ on $[0, \beta]$ as $k \to \infty$ since $0 < e_k \leq \frac{1}{E}$, $\frac{e_k}{p^*(k) - e_k} \to 0$ and $(\frac{x}{k} - 1)^2$ converges uniformly to 1 on $[0, \beta]$ as $k \to \infty$.

Next, let

$$\eta = \min\left( \min_{x \in [0,\beta]} f(x),\ \min_{x \in [0,\beta]} \frac{1}{p^*(x)},\ \frac{E}{4}\right) > 0.$$

Select $\mu \geq \beta$ such that $k \geq \mu$ implies that $\sqrt{k} \geq \beta$, $k > 1$, $\max\{\left|\frac{1}{p_k(x, e_k)} - \frac{1}{p^*(x)}\right|: x \in [0, \beta]\} \leq \frac{\eta}{2}$. Thus, for $k \geq \mu$, $\frac{1}{p_k(x, e_k)} \geq \frac{\eta}{2} > 0$, for all $x \in [0, \beta]$. This implies that for $k \geq \mu$, $\frac{1}{p_k(x, e_k)}$ is positive and converges uniformly to $\frac{1}{p^*(x)}$ on $[0, \beta]$. In addition, for $k \geq \mu$ and $x \in [\beta, \sqrt{k}]$ we have $p_k(x, e_k) \geq e_k + (p^*(x) - e_k)[(\frac{1}{\sqrt{k}} - 1)^2]$ $\geq e_k + \frac{1}{2}(p^*(x) - e_k) \geq \frac{1}{2}p^*(x) \geq \frac{4}{E}$ as $p^*(x) \geq \frac{8}{E}$ for $x \geq \beta$. Since

$\max\{\frac{1}{p_k(x, e_k)}: x \in [\beta, 2k]\} = E$ we have that if $t_k \in [\beta, 2k]$ is such that $\frac{1}{p_k(t_k, e_k)} = E$ then $t_k > \sqrt{k}$ for $k \geq \mu$.

Next, note that for $x \geq k \geq \mu$, $p_k(x, e_k)$ is a monotone increasing function of x and that $p_k(2k, e_k) = e_k + (p^*(2k) - e_k)(1 + \frac{e_k}{p^*(2k) - e_k})$ $\geq p^*(2k) \geq \frac{8}{E}$. Thus, $\frac{1}{p_k(x, e_k)} \leq \frac{E}{8}$ for $x \geq 2k$. Summarizing, we have shown that $\frac{1}{p_k(x, e_k)} \leq \frac{E}{4}$ for $x \in [0, \sqrt{k}]$, $\frac{1}{p_k(x, e_k)} \leq \frac{E}{8}$ for $x \geq 2k$ and $\frac{1}{p_k(x, e_k)} \leq E$ for $x \in [\sqrt{k}, 2k]$ with $t_k \in [\sqrt{k}, 2k]$ a point at which the value E is attained.

Next, define $\alpha_k$ by $E - \alpha_k = \max\{(\frac{1}{p_k(x, e_k)} - f(x)): x \in [\beta, 2k]\}$. Since $f(x) \leq \frac{E}{8}$ for $x \geq \beta$ and $\frac{1}{p_k(t_k, e_k)} = E$ we have that $E - \alpha_k \geq E - f(t_k)$ $\geq \frac{7}{8}E$ implying that $\frac{E}{8} \geq f(t_k) \geq \alpha_k$. Let $y_k \in [\beta, 2k]$ be such that $\frac{1}{p_k(y_k, e_k)} - f(y_k) = E - \alpha_k$ for each $k \geq \mu$. Since $\frac{1}{p_k(x, e_k)} \leq \frac{E}{8}$ for $x \in [\beta, \sqrt{k}]$ we have that $y_k \in [\sqrt{k}, 2k]$. Also, since $f(t_k) \to 0$ as $k \to \infty$ (as $t_k \to \infty$) it follows that $\alpha_k \to 0$ as $k \to \infty$. Noting that $f(x) \leq \frac{E}{8}$ for $x \in [\beta, \infty)$ and that $\frac{1}{p_k(x, e_k)} \leq \frac{E}{8}$ for $x \geq 2k$ we have that $\left|f(x) - \frac{1}{p_k(x, e_k)}\right|$ $\leq E - \alpha_k$ for $x \in [\beta, \infty)$ and $k \geq \mu$. Also, since $\frac{1}{p_k(t_k, e_k)} = E$ and $\frac{1}{p^*(t_k)} \leq \frac{E}{8}$ we have that $\|\frac{1}{p_k} - \frac{1}{p^*}\| \geq \left|\frac{1}{p_k(t_k, e_k)} - \frac{1}{p^*(t_k)}\right| \geq \frac{7E}{8}$.

Now define $g_k$ by (for $k \geq \mu$)

$$g_k(x) = \begin{cases} f(x) + \dfrac{1}{p_k(x, e_k)} - \dfrac{1}{p^*(x)}, \ x \in [0, \beta], \ \left|f(x) - \dfrac{1}{p^*(x)}\right| \leq E - \alpha_k \\ E - \alpha_k + \dfrac{1}{p_k(x, e_k)}, \ x \in [0, \beta], \ f(x) - \dfrac{1}{p^*(x)} > E - \alpha_k \\ -E + \alpha_k + \dfrac{1}{p_k(x, e_k)}, \ x \in [0, \beta], \ f(x) - \dfrac{1}{p^*(x)} < -E + \alpha_k \\ f(x), \qquad x \geq \beta + \dfrac{1}{k} \\ \text{linear on } [\beta, \beta + \frac{1}{k}] \text{ with endpoint values } f(\beta) + \dfrac{1}{p_k(\beta, e_k)} - \dfrac{1}{p^*(\beta)} \\ \text{and } f(\beta + \frac{1}{k}). \end{cases}$$

Observe that $g_k(x) > 0$ for all $x \geq 0$. Indeed, for $x \in [0, \beta]$ with $\left|f(x) - \frac{1}{p^*(x)}\right| \leq E - \alpha_k$ we have that $g_k = f(x) + \frac{1}{p_k(x, e_k)} - \frac{1}{p^*(x)} \geq f(x) - \frac{\eta}{2} \geq \frac{\eta}{2} > 0$. For $x \in [0, \beta]$ with $f(x) - \frac{1}{p^*(x)} > E - \alpha_k$, $g_k(x) = E - \alpha_k + \frac{1}{p_k(x, e_k)} \geq \frac{7E}{8} + \frac{1}{p_k(x, e_k)} > 0$ and for $x \in [0, \beta]$ with $f(x) - \frac{1}{p^*(x)} < -E + \alpha_k$,

$g_k(x) = -E + \alpha_k + \frac{1}{p_k(x, e_k)} = -E + \alpha_k + \frac{1}{p^*(x)} + \frac{1}{p_k(x, e_k)} - \frac{1}{p^*(x)} \geq f(x) - \frac{\eta}{2} \geq \frac{\eta}{2} > 0$. Since $f(\beta) + \frac{1}{p_k(\beta, e_k)} - \frac{1}{p^*(\beta)} \geq f(\beta) - \frac{\eta}{2} \geq \frac{\eta}{2} > 0$ and $f(\beta + \frac{1}{k}) > 0$ we have that $g_k(x) > 0$ on $[\beta, \beta + \frac{1}{k}]$ and finally $g_k$ is positive on $[\beta + \frac{1}{k}, \infty)$ as $f$ is. To see that $g_k(x)$ is continuous on $[0, \infty)$ one must only check on $[0, \beta]$ as for $x > \beta$ it is clearly continuous. However, on $[0, \beta]$, $g_k(x)$ is simply the truncation of $f(x) - \frac{1}{p^*(x)}$ to the range $[-E + \alpha_k, E - \alpha_k]$ plus the continuous function $\frac{1}{p_k(x, e_k)}$ showing that $g_k \in C_0^+[0, \infty)$.

Next, let us consider $\left|g_k(x) - \frac{1}{p_k(x, e_k)}\right|$. Note that by construction $\left|g_k(x) - \frac{1}{p_k(x, e_k)}\right| \leq E - \alpha_k$ for $x \in [0, \beta]$ and that, if $\{x_i\}_{i=0}^{n}$ with $x_0 < x_1 < \ldots < x_n$ is a nonstandard alternating set for $f - \frac{1}{p^*}$ then we must have that $x_n < \beta$ and

$$g_k(x_i) - \frac{1}{p_k(x_i, e_k)} = \operatorname{sgn}(f(x_i) - \frac{1}{p^*(x_i)})(E - \alpha_k) = (-1)^{n-i}(E - \alpha_k).$$

Next, on $[\beta, \beta + \frac{1}{k}]$ we have that $f(\beta) \leq \frac{E}{8}$, $f(\beta + \frac{1}{k}) \leq \frac{E}{8}$ and $\left|\frac{1}{p_k(\beta, e_k)} - \frac{1}{p^*(\beta)}\right| \leq \frac{E}{8}$ so that $g_k(\beta) \leq \frac{E}{4}$ and $g_k(\beta + \frac{1}{k}) \leq \frac{E}{8}$. Thus, $g_k(x) \leq \frac{E}{4}$ on $[\beta, \beta + \frac{1}{k}]$. Also, recall that $\frac{1}{p_k(x, e_k)} \leq \frac{E}{4}$ on $[0, \sqrt{k}]$ so that $\left|g_k(x) - \frac{1}{p_k(x, e_k)}\right| \leq \frac{E}{4}$ on $[\beta, \beta + \frac{1}{k}]$. Finally, we noted earlier that $\left|f(x) - \frac{1}{p_k(x, e_k)}\right| \leq E - \alpha_k$ on $[\beta, \infty)$ so that $\left|g_k(x) - \frac{1}{p_k(x, e_k)}\right| \leq E - \alpha_k$ on $[0, \infty)$. Since there exists $y_k \in [\sqrt{k}, 2k]$ at which $f(y_k) - \frac{1}{p_k(y_k, e_k)} = -(E - \alpha_k)$ we have that $g_k - \frac{1}{p_k}$ possesses a standard alternating set at the points $x_0 < x_1 < \ldots < x_n < y_k$ and thus $\frac{1}{p_k}$ is the best approximation to $g_k$ from $R_n$ on $[0, \infty)$. Finally, it is a straightforward argument to prove that $g_k$ converges uniformly to f. Thus, once again reindexing the sequence $\{\frac{1}{p_k}\}_{k=\mu}^{\infty}$ gives the desired result.

∎

Next, we wish to show that if $f \in C_0^+[0, \infty)$ has $\frac{1}{p^*}$ as its best approximation from $R_n$ with $\partial p^* = n - 1$ then the best approximation operator is continuous. This we do in the following theorem.

<u>THEOREM 4</u>. Let $f \in C_0^+[0, \infty) \sim R_n$ and let $\frac{1}{p^*}$ be its best approximation from $R_n$ on $[0, \infty)$ with $\partial p^* = n - 1$. Then, the best approximation operator is continuous at f.

<u>Proof</u>. Let $\{g_k\}_{k=1}^{\infty} \subset C_0^+[0, \infty)$ with $g_k \to f$ uniformly on $[0, \infty)$. Further, let $\frac{1}{p_k} \in R_n$ be the best approximation to $g_k$ on $[0, \infty)$ for each k. Then, we must prove that $\|\frac{1}{p_k} - \frac{1}{p^*}\| \to 0$ as $k \to \infty$. Let us first note that $\|g_k - \frac{1}{p_k}\| \leq \|g_k - \frac{1}{p^*}\|$ implying that $\limsup_{k\to\infty} \|g_k - \frac{1}{p_k}\| \leq \limsup_{k\to\infty} \|g_k - \frac{1}{p^*}\| = \|f - \frac{1}{p^*}\| = E$. Also, $E = \|f - \frac{1}{p^*}\| \leq \|f - \frac{1}{p_k}\| \leq \|f - g_k\| + \|g_k - \frac{1}{p_k}\|$ implying that $E = \liminf_{k\to\infty}(E - \|f - g_k\|) \leq \liminf_{k\to\infty} \|g_k - \frac{1}{p_k}\|$. Combining these results gives that $\lim_{k\to\infty} \|g_k - \frac{1}{p_k}\| = E$. In addition, since $E \leq \|f - \frac{1}{p_k}\| \leq \|f - g_k\| + \|g_k - \frac{1}{p_k}\|$ we also have that $\lim_{k\to\infty} \|f - \frac{1}{p_k}\| = E$.

Next, fix $y \in [0, \infty)$ such that $f(y) = \max\{f(x): x \in [0, \infty)\}$. Then since a constant cannot be a best approximation to f from $R_n$ on $[0, \infty)$ we must have that $2E < f(y)$. Select $\delta > 0$ such that for $x \in I = [y-\delta, y+\delta] \cap [0,\infty)$ we must have $f(x) \geq \frac{1}{2}(2E + f(y)) > 2E$. Choose $\beta$ such that $k \geq \beta$ implies that $\|f - \frac{1}{p_k}\| \leq \frac{3}{2}E$. Then for $k \geq \beta$ and $x \in I$, we have that

$$0 < m = 2E - \frac{3}{2}E \leq f(x) - \frac{3}{2}E \leq \frac{1}{p_k(x)} \leq f(x) + \frac{3}{2}E \leq \|f\| + \frac{3}{2}E = M.$$

In addition, observe that the inequality $\frac{1}{p_k(x)} \leq M$ holds for all $x \in [0, \infty)$ and $k \geq \beta$. Let $\{p_\nu\}$ be a subsequence of $\{p_k\}$. Then, since $\frac{1}{M} \leq p_\nu(x) \leq \frac{1}{m}$

for all $x \in I$, there exists a subsequence $\{p_\mu\}$ of $\{p_\nu\}$ such that $p_\mu$ converges uniformly to some $\bar{p} \in \Pi_n$ on I. This implies that the coefficients of $p_\mu$ converge to the coefficients of $\bar{p}$ which in turn implies that for each $x \in [0, \infty)$, $p_\mu(x) \to \bar{p}(x)$. Thus, we must have $\frac{1}{M} \leq \bar{p}(x) \leq \frac{1}{m}$ on I and $\frac{1}{M} \leq \bar{p}(x)$ on $[0, \infty)$. This last inequality shows that $\bar{p} \in R_n$. Furthermore, for $x \in [0, \infty)$ fixed, $\left|f(x) - \frac{1}{\bar{p}(x)}\right|$ $= \lim_{n\to\infty}\left|f(x) - \frac{1}{p_\mu(x)}\right| \leq \lim_{n\to\infty}\left\|f - \frac{1}{p_\mu}\right\| = E$. Thus, $\left\|f - \frac{1}{\bar{p}}\right\| \leq E$ implying that $\bar{p} \equiv p^*$ by the uniqueness of best approximations from $R_n$. Since this is true for any subsequence $\{p_\nu\}$ of $\{p_k\}$ we must have that this is also true for the full sequence $\{p_k\}$. That is, that $p_k$ converges uniformly to $p^*$ on I and pointwise on $[0, \infty)$. To complete this argument we must prove that $\frac{1}{p_k}$ converges uniformly to $\frac{1}{p^*}$ on $[0, \infty)$. From the above discussion we have that $\frac{1}{p_k}$ converges pointwise to $\frac{1}{p^*}$ on $[0, \infty)$ and, in fact, on any fixed closed interval $[0, \alpha]$, $\alpha > 0$, $\frac{1}{p_k}$ converges uniformly to $p^*$ (due to the coefficient convergence).

In order to establish this final fact, we must examine the coefficient convergence in more detail. Thus, let $p^*(x)=a^*_{n-1}x^{n-1}+\ldots+a^*_0$ with $a^*_{n-1} > 0$ (here we are using our hypothesis that $\partial p^* = n - 1$ and $p^* \in R_n$) and let $p_k(x) = a^k_n x^n + \ldots + a^k_0$ where we know that the leading nonzero coefficient of $p_k$ must be positive. In addition, we have that $a^k_j \to a^*_j$ as $k \to \infty$ for $j = 0, 1, \ldots, n$ where $a^*_n = 0$. Thus, there exists $\gamma \geq \beta$ such that $k \geq \gamma$ implies that $a^k_{n-1} \geq \frac{a^*_{n-1}}{2} > 0$ and $|a^k_j - a^*_j| \leq 1$ for $j = 0, \ldots, n - 2$. Thus, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$\hat{p}(x) = \frac{a^*_{n-1}}{2}x^{n-1} + (a^*_{n-2} - 1)x^{n-2} + \ldots + (a^*_0 - 1) \geq \frac{2}{\varepsilon}$. Since $p_k(x) \geq \hat{p}(x)$ for $k \geq \gamma$ and $p^*(x) > \hat{p}(x)$ for all $x \geq \delta$ we have that

$$\left|\frac{1}{p_k(x)} - \frac{1}{p^*(x)}\right| \leq \left|\frac{1}{p_k(x)}\right| + \left|\frac{1}{p^*(x)}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $k \geq \gamma$ and $x \geq \delta$. On $[0, \delta]$ we have that $\frac{1}{p_k}$ converges uniformly to $\frac{1}{p^*}$. Thus, we may select $K \geq \gamma$ such that $k \geq K$ implies $\left|\frac{1}{p_k(x)} - \frac{1}{p^*(x)}\right| < \varepsilon$ for all $x \in [0, \delta]$. Hence, for $k \geq K$ we have that $\|\frac{1}{p_k} - \frac{1}{p^*}\| < \varepsilon$ implying the desired result.

∎

## Concluding Remarks

Observe that the question of whether or not a strong uniqueness result holds for the case that $f \in C_0^+[0, \infty)$ with its best approximation $\frac{1}{p^*}$ from $R_n$ satisfying $\partial p^* = n-1$ remains open. Likewise, the question of Lipschitz continuity of the best approximation operator remains open in this case.

A second item of interest is that in ordinary rational approximation on a finite interval, nonstandard (i.e., fewer) alternation due to degeneracy of the best approximation may be unimportant as the set of f with degenerate best approximations is nowhere dense [5, 7]. If the corresponding result that $\{f:$ the best approximation $\frac{1}{p^*} \in R_n$ has $\partial p^* < n\}$ was nowhere dense then we could expect to be able to usually employ the simpler theory of [8] for this problem. However, the continuity result for degree n - 1 implies that every f with nonstandard alternation and best approximation $\frac{1}{p^*} \in R_n$ with $\partial p^* = n - 1$ has all g sufficiently close with nonstandard alternation and best approximations of degree n - 1. In this regard, an interesting question is to characterize those f for which nonstandard alternation will occur. Some initial results in this direction have been obtained by the second author and D. Leeming.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|---|
| 1. REPORT NUMBER 19<br>18 AFOSR-TR-79-0856 | 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle)<br>6 CONTINUITY OF BEST RECIPROCAL POLYNOMIAL APPROXIMATION ON [0, oo) | | 5. TYPE OF REPORT & PERIOD COVERED<br>9 Interim rept. |
| | | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s)<br>10 Charles B. Dunham ~~and~~ G. D. Taylor | | 8. CONTRACT OR GRANT NUMBER(s)<br>15 AFOSR-76-2878, NSF-MCS78-05847 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>Colorado State University<br>Department of Mathematics<br>Fort Collins, CO 80523 (0, infinity) | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS<br>61102F 16 2304 17 A3 |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>Air Force Office of Scientific Research/NM<br>Bolling AFB, D. C. 20332 | | 12. REPORT DATE<br>11 April 1979 |
| | | 13. NUMBER OF PAGES<br>14 |
| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)<br>12 15p. | | 15. SECURITY CLASS. (of this report)<br>UNCLASSIFIED |
| | | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Reciprocal polynomial approximation on $[0, \infty)$, constrained rational approximation.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)


Continuity of the best approximation operator is studied for approximating continuous positive functions tending to zero as $x \to \infty$ by reciprocals of polynomils on $[0, \infty)$ is studied. Both negative and positive results are obtained.

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